Existence of extremal functions for a family of Caffarelli-Kohn-Nirenberg inequalities*

X. Zhong & W. Zou

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Abstract

Consider the following inequalities due to Caffarelli, Kohn and Nirenberg *(Compositio Mathematica,1984):*

$$\Big(\int_\Omega \frac{|u|^r}{|x|^s}dx\Big)^{\frac{1}{r}} \leq C(p,q,r,\mu,\sigma,s)\Big(\int_\Omega \frac{|\nabla u|^p}{|x|^\mu}dx\Big)^{\frac{a}{p}}\Big(\int_\Omega \frac{|u|^q}{|x|^\sigma}dx\Big)^{\frac{1-a}{q}},$$

where $\Omega\subset\mathbb{R}^N (N\geq 2)$ is an open set; p,q,r,μ,σ,s,a are some parameters satisfying some balanced conditions. When Ω is a cone in $\mathbb{R}^N$ (for example, $\Omega=\mathbb{R}^N$), we prove the sharp constant $C(p,q,r,\mu,\sigma,s)$ can be achieved for a very large parameter space. Besides, we find some sufficient conditions which guarantee that the following Sobolev spaces

$$W^{1,p}_\mu(\Omega),\ W^{1,p}_\mu(\Omega)\cap L^p(\Omega),\ H^{1,p}(\mathbb{R}^N)$$

are compactly embedded into $L^r(\mathbb{R}^N, \frac{dx}{|x|^s})$ for some new ranges of parameters, where $W^{1,p}_\mu(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\Big(\int_\Omega \frac{|\nabla u|^p}{|x|^\mu}dx\Big)^{\frac{1}{p}}$. As applications, we also study the equation

$$-div\Big(\frac{|\nabla u|^{p-2}\nabla u}{|x|^\mu}\Big)=\lambda V(x)|u|^{q-2}u,\quad u\in W^{1,p}_\mu(\Omega)$$

under some proper conditions on $V(x)$.

Key words: Rellich-Kondrachov theorem, Caffarelli-Kohn-Nirenberg inequality, Ground state, Extremal functions.

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1 Introduction

In 1984, Caffarelli, Kohn and Nirenberg established a family of interpolation inequalities, nowadays called Caffarelli-Kohn-Nirenberg (CKN) inequalities.

Theorem A(cf.[6]) *Assume that $p, q, r, \alpha, \beta, \sigma$ and a are fixed real numbers (called parameters) satisfying*

$$p \geq 1, \quad q \geq 1, \quad r > 0, \quad 0 \leq a \leq 1; \tag{1.1}$$

$$\frac{1}{p} + \frac{\alpha}{N} > 0, \quad \frac{1}{q} + \frac{\beta}{N} > 0, \quad \frac{1}{r} + \frac{\gamma}{N} > 0, \tag{1.2}$$

where $\gamma = a\sigma + (1-a)\beta$. Then there exists a positive constant C such that the following inequality holds

$$\big||x|^\gamma u\big|_{L^r} \leq C\big||x|^\alpha |Du|\big|_{L^p}^a \big||x|^\beta u\big|_{L^q}^{1-a}, \quad \forall u \in C_0^\infty(\mathbb{R}^N) \tag{1.3}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{N} = a\big(\frac{1}{p} + \frac{\alpha - 1}{N}\big) + (1-a)\big(\frac{1}{q} + \frac{\beta}{N}\big) \tag{1.4}$$

(this is dimensional balance)

$$0 \leq \alpha - \sigma \quad \textit{if } a > 0,$$

and

$$\alpha - \sigma \leq 1 \quad \textit{if } a > 0 \textit{ and } \frac{1}{p} + \frac{\alpha - 1}{N} = \frac{1}{r} + \frac{\gamma}{N}.$$

Furthermore, on any compact set in parameter space in which (1.1),(1.2),(1.4) and $0 \leq \alpha - \sigma \leq 1$ hold, the constant C is bounded.

□

Some variant versions of the CKN inequality with higher order derivatives were given by Lin [17]. Note that the CKN inequality and its variance include many well-known inequalities such as the Hardy-Sobolev inequality, Gagliardo-Nirenberg inequality, etc. They play a crucial role in the elliptic partial differential equations. Recall a version of the Gagliardo-Nirenberg inequality

$$|u|_r \leq C|\nabla u|_2^a |u|_2^{1-a}. \tag{1.5}$$

When $2 < r < 2^* := \frac{2N}{N-2}$ $(N \geq 3)$, the dimensional balance condition implies that $0 < a < 1$. Then by the Young inequality and Sobolev inequality, we see that $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is an continuous embedding for $2 \leq r \leq 2^*$ which has been wildly used now. If we consider that $a = 1$ in (1.3), then we have the following inequality without interpolation:

$$\big\||x|^\gamma u\big\|_{L^r} \leq C\big\||x|^\alpha |Du|\big\|_{L^p} \text{ for all } u \in C_0^\infty(\mathbb{R}^N), \tag{1.6}$$

where $p \geq 1, \frac{1}{r} + \frac{\gamma}{N} > 0, r \begin{cases} \leq p^* := \frac{Np}{N-p}, & \text{if } N > p, \\ < \infty, & \text{if } N \leq p, \end{cases}$ and

$$\frac{1}{r} + \frac{\gamma}{N} = \frac{1}{p} + \frac{\alpha - 1}{N}, \tag{1.7}$$

which is dimensional balance condition. We call (1.6) the general Hardy-Sobolev inequality since when $\gamma = \alpha = 0$, (1.6) returns to the classical Sobolev inequality:

$$|u|_r \leq C|\nabla u|_p \text{ with } r = p^*, N > p, \tag{1.8}$$

When $\alpha = -t, \gamma = -t - 1$, (1.6) becomes

$$\left|\frac{u}{|x|^{t+1}}\right|_p \leq C\left|\frac{\nabla u}{|x|^t}\right|_p, \quad N - p - pt > 0, \tag{1.9}$$

which is called the general weighted Hardy inequality.

Much progress has been made on (1.6) for the case of $p = 2$. For example, in [1, 24], Aubin and Talenti gave the best constant and the minimizers for the Sobolev inequality (1.8) via the Schwarz symmetrization and the Bliss inequality in [5]. In [16], Lieb applied the same type of symmetrization to study (1.6) with $\alpha = 0, p = 2, -1 < \gamma < 0$. The results of [16] had been generalized by Chou and Chu in [8] to the case of $\alpha - 1 < \gamma \leq \alpha \leq 0, p = 2$. A further generalization was given by Wang and Willem in [25] for the case of $p = 2$. When $p = 2$ and $\alpha > 0$, it was also studied in the papers [7]. For the case of $p \neq 2$ but with different geometries of the domain $\Omega \subset \mathbb{R}^N$, we refer to [2]. More results about the related progress, we refer to [26, 18, 14, 13, 10]. We remark that the papers mentioned in this paragraph mainly deal with the inequality (1.6) without interpolation term.

When $a \neq 1$, the CKN inequality involves three terms (i.e., interpolation), which make the problem much tough and there are rare paper investigating this case, we just find the following partial answers (see the review paper by Dolbeault and Esteban [11]):

- When $\alpha = \beta = \gamma = 0, p = 2, q = p + 1$ and $r = 2p$. For such a very special case, the sharp constant and the extremal functions of inequality (1.3) are given by Del Pino and Dolbeault [9].
- When $p = q = 2, -\frac{N-2}{2} < \alpha, \beta = \alpha - 1, \alpha - 1 \leq \gamma < \alpha$, and $r = \frac{2N}{N+2(\gamma-\alpha)}$. Under these assumptions, together with a special region of a and other conditions, the sharp constant and extremal functions of the CKN inequality (1.3) are studied by Dolbeault, Esteban, Tarantello and Tertikas [15], Dolbeault and Esteban [12].

In the current paper, we consider the general cases of the CKN inequality: $p > 1$ and it has interpolation term.

We make a transformation first. Let $\alpha = -\frac{\mu}{p}, \beta = -\frac{\sigma}{q}, \gamma = -\frac{s}{r}$ in (1.3), then a direct computation shows that $a = \frac{[(N-\sigma)r-(N-s)q]p}{[(N-\sigma)p-(N-\mu-p)q]r}$. We obtain the following

version of the CKN inequality:

$$\Big(\int_{\Omega}\frac{|u|^r}{|x|^s}dx\Big)^{\frac{1}{r}}\leq C(p,q,r,\mu,\sigma,s)\Big(\int_{\Omega}\frac{|\nabla u|^p}{|x|^{\mu}}dx\Big)^{\frac{a}{p}}\Big(\int_{\Omega}\frac{|u|^q}{|x|^{\sigma}}dx\Big)^{\frac{1-a}{q}}, \tag{1.10}$$

In present paper, when Ω is a cone (i.e., $\lambda x\in\Omega$ for all $x\in\Omega$ and $\lambda>0$), we can obtain the existence of extremal functions for the CKN inequalities (1.10). Define

$$p^*(s,\mu):=\frac{p(N-s)}{N-p-\mu}.$$

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^N$ be a cone (in particular, $\Omega=\mathbb{R}^N$). Assume that $p>1, s>0,\ \max\{\sigma,s\}<\mu+p<N$, $1\leq r, 1\leq q<\min\{p^*,p^*(\sigma,\mu)\}$, $\max\{\dfrac{p(\sigma-s)}{N-\mu-p}+q, \dfrac{\sigma-s}{N-\sigma}q+q\}<r<\min\{p^*,p^*(s,\mu)\}$ and*

$$\begin{cases}p(s-\sigma)+q(\mu+p-s)<r(\mu+p-\sigma)\\ (Np-Nr+pr)(s-\sigma)>(N\mu-Ns+ps)(r-q)\end{cases}, \tag{1.11}$$

then the sharp constant $C(p,q,r,\mu,\sigma,s)$ can be achieved and

$$C(p,q,r,\mu,\sigma,s)=(\frac{1}{\rho})^{\frac{(\mu+p-\sigma)r+(p-q)(N-s)}{r[(N-\sigma)p-(N-\mu-p)q]}},$$

where

$$\rho:=\inf\Big\{\int_{\Omega}\frac{|\nabla u|^p}{|x|^{\mu}}dx+\lambda^*\int_{\Omega}\frac{|u|^q}{|x|^{\sigma}}dx\ :\ \int_{\Omega}\frac{|u|^r}{|x|^s}dx=1\Big\} \tag{1.12}$$

which can be attained and

$$\begin{aligned}\lambda^*:=&\Big\{\frac{p(N-s)-(N-\mu-p)r}{(\mu+p-\sigma)r+(p-q)(N-s)}\Big\}^{\frac{(\mu+p-\sigma)r+(p-q)(N-s)}{(N-s)p-(N-\mu-p)r}}\\ &\cdot\Big\{\frac{(N-\sigma)r-(N-s)q}{p(N-s)-(N-\mu-p)r}\Big\}^{\frac{(N-\sigma)r-(N-s)q}{p(N-s)-(N-\mu-p)r}}.\end{aligned} \tag{1.13}$$

Remark 1.1. *When $\sigma=0, 1<p=q<N$, each of the the following conditions meets the hypotheses of Theorem 1.1:*

(1) $\mu=0, 0<s<p<N, p<r<\frac{p(N-s)}{N-p}$;

(2) $\mu>0, \frac{N\mu(r-p)}{p^2}<s<\mu+p<N, p<r<\min\{\frac{pN}{N-p},\frac{p(N-s)}{N-\mu-p}\}$;

(3) $\mu<0, 0<s<\mu+p<N, p<r<\frac{p(N-s)}{N-\mu-p}$.

In fact, under these conditions we shall show that the embedding

$$W^{1,p}_{\mu}(\Omega)\cap L^p(\Omega)\hookrightarrow L^r(\Omega,\frac{dx}{|x|^s})$$

is a compact embedding, where we denote by $W_\mu^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \Big(\int_\Omega \frac{|\nabla u|^p}{|x|^\mu} dx\Big)^{\frac{1}{p}} \tag{1.14}$$

and $L^r(\Omega, \frac{dx}{|x|^s})$ stands for the completion of $C_0^\infty(\Omega)$ with respect to the norm of

$$|u|_{r,s} := \Big(\int_\Omega \frac{|u|^r}{|x|^s} dx\Big)^{\frac{1}{r}}.$$

See Corollary 3.1 in Section 3.

Remark 1.2. *It is well known that $H^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is a continuous embedding for $r \in [p, p^*)$ but not compact. However, we will prove that $H^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N, \frac{dx}{|x|^s})$ is a compact embedding for $s > 0$ and $r \in [p, p^*(s))$. See Remark 3.1.*

In this paper, we also study the following problem

$$-\, div\Big(\frac{|\nabla u|^{p-2}\nabla u}{|x|^\mu}\Big) = \lambda V(x)|u|^{q-2}u, \ \ u \in W_\mu^{1,p}(\Omega), \tag{1.15}$$

where $1 < q < p^* := \frac{Np}{N-p}, 1 < p < N, \mu + p < N$ and $\Omega \subset \mathbb{R}^N$ is an open Lipschitz domain and λ is a parameter. We assume that $V \in L^1_{loc}(\Omega), V = V_+ - V_-$, where $V_\pm(x) := \max\{\pm V(x), 0\}$. Our basic assumption is

(**H**) $V \in L^1_{loc}(\Omega), V_+ = V_1 + V_2 \neq 0$, there exists some $\frac{N\mu}{N-p} \leq \eta < \min\{\mu + p, N - \frac{q}{p}(N-\mu-p)\}$ such that $\big|V_1(x)\big|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}} |x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} \in L^1(\Omega)$ and one of the following holds

(H_1) $1 < q < p, \Omega$ is bounded and $\sup_{y\in\bar{\Omega}} \lim_{\substack{x\to y\\ x\in\Omega}} |x-y|^{\mu+p} V_2(x) < \infty$.

(H_2) $1 < q < p, \Omega$ is unbounded (in particular, $\Omega = \mathbb{R}^N$),

$$\sup_{y\in\overline{\Omega\cap B_R(0)}} \lim_{\substack{x\to y\\ x\in\Omega}} |x-y|^{\mu+p} V_2(x) < \infty \text{ for any fixed } R > 0$$

and

$$\lim_{R\to\infty} \int_{\{x\in\Omega: |x|>R\}} \big(V_2(x)\big)^{\frac{p}{p-q}} |x|^{\frac{(\mu+p)q}{p-q}} dx \to 0.$$

(H_3) $p \leq q$. For any $y \in \bar{\Omega}, \lim_{\substack{x\to y\\ x\in\Omega}} |x-y|^{\bar\sigma} V_2(x) = 0$ and $\lim_{\substack{|x|\to\infty\\ x\in\Omega}} |x|^{\bar\sigma} V_2(x) = 0$, where

$$\bar\sigma := N - \frac{q}{p}(N-\mu-p) \in \Big(\frac{N\mu}{N-p}, \mu + p\Big].$$

Here comes our another main theorem:

Theorem 1.2. *Assume* $(\mathbf{H})$.

(1) *If $q = p$, then the equation (1.15) has a sequence of eigenfunctions $\{\varphi_n\}$, the corresponding eigenvalues $\{\lambda_n\}$ satisfying $\lambda_n \to \infty$ as $n \to \infty$.*

(2) *If $q > p$, then for any positive fixed λ,* (1.15) *possesses a sequence of solutions $\{v_n\}$ such that $0 < c_1 \le c_2 \le \cdots \le c_n \to \infty$ as $n \to \infty$.*

(3) *If $q < p$, then for any positive fixed λ,* (1.15) *possesses a sequence of solutions $\{v_n\}$ such that $-\infty < c_1 \le c_2 \le \cdots \le c_n \to 0$ as $n \to \infty$.*

Where

$$c_n := \Phi(v_n) := \frac{1}{p}\int_\Omega \frac{|\nabla v_n|^p}{|x|^\mu}dx - \frac{1}{q}\lambda\int_\Omega V(x)|v_n|^q dx.$$

Remark 1.3. *When $\mu = 0$ and $q = p$, Theorem 1.2 was established by Szulkin-Willem [23].*

2 Preliminaries

Firstly, based on the original Rellich-Kondrachov theorem and the general weighted Hardy inequality (1.9), we obtain the following result through a transformation.

Lemma 2.1. *Assume that $1 \le p < N, N - p - \mu > 0$ and that $\{u_n\} \subset W^{1,p}_\mu(\Omega)$ is bounded. Then there exists some $u \in W^{1,p}_\mu(\Omega)$ and, up to a subsequence, $u_n \to u$ a.e. in Ω.*

Proof. For the case of $\mu = 0$, it can be easily obtained by the original Rellich-Kondrachov theorem and the diagonal trick. Next, we only consdier that $\mu \neq 0$. Denote $\bar\mu := \frac{\mu}{p}$, note that

$$\nabla\big(\frac{u_n}{|x|^{\bar\mu}}\big) = \frac{\nabla u_n}{|x|^{\bar\mu}} - \bar\mu|x|^{-\bar\mu-2}xu_n,$$

it follows that

$$\big|\nabla(\frac{u_n}{|x|^{\bar\mu}})\big| \le \frac{|\nabla u_n|}{|x|^{\bar\mu}} + |\bar\mu|\frac{|u_n|}{|x|^{\bar\mu+1}}.$$

Recalling that for $p > 0$, $|x+y|^p \le \max\{1, 2^{p-1}\}(|x|^p + |y|^p)$. Hence, combining with the general weighted Hardy inequality (1.9) due to the fact of $N - p - \mu > 0$, we have

$$\begin{aligned}
&\int_\Omega \big|\nabla(\frac{u_n}{|x|^{\bar\mu}})\big|^p dx \\
&\le 2^{p-1}\Big[\int_\Omega \big|\frac{|\nabla u_n|}{|x|^{\bar\mu}}\big|^p dx + |\bar\mu|^p\int_\Omega \big|\frac{|u_n|}{|x|^{\bar\mu+1}}\big|^p dx\Big] \\
&\le C(p,\mu)\int_\Omega \big|\frac{|\nabla u_n|}{|x|^{\bar\mu}}\big|^p dx = C(p,\mu)\int_\Omega \frac{|\nabla u_n|^p}{|x|^\mu}dx. \qquad (2.1)
\end{aligned}$$

Thus, $\{\frac{u_n}{|x|^{\bar\mu}}\}$ is a bounded sequence in $W^{1,p}(\Omega)$. It follows from the well-known Rellich-Kondrachov compactness theorem and the standard diagonal trick, we obtain

that, up to a subsequence, $\dfrac{u_n}{|x|^{\bar{\mu}}} \to \dfrac{u}{|x|^{\bar{\mu}}}$ a.e. in Ω. Then it is natural to see that $u_n \to u$ a.e. in Ω. It follows from the Fatou's Lemma that $u \in W^{1,p}_{\mu}(\Omega)$. $\square$

Now, we can prove the weighted Rellich-Kondrachov compactness theorem:

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^N$ is a bounded open Lipschitz domain. Suppose that $1 \le p < N, -\infty < \mu < N-p$, then the embedding*

$$W^{1,p}_{\mu}(\Omega) \hookrightarrow L^q(\Omega, \frac{dx}{|x|^s})$$

is compact if

$$\begin{cases} \frac{N\mu}{N-p} \le s \le \mu + p, \\ 1 \le q < p^*(s,\mu), \end{cases} \quad \text{or} \quad \begin{cases} s < \frac{N\mu}{N-p}, \\ 1 \le q < p^*. \end{cases}$$

Moreover, if $s \ge \max\{0, \frac{N\mu}{N-p}\}$, the conclusion is still valid when domain Ω is unbounded but with finite Lebesgue measure.

Proof. We firstly consider the case that Ω is bounded. Assume that $\sup\limits_n \|u_n\| < \infty$. By Lemma 2.1, without loss of generality, we may assume that $u_n \to u$ a.e. in Ω for some $u \in W^{1,p}_{\mu}(\Omega)$. Since $s < N$ and Ω is bounded, it is easy to see that $\nu(\Omega) < \infty$, where the new measure $d\nu := \frac{dx}{|x|^s}$ and $\nu\big|_{\Omega}$ is absolutely continuous with respect to the usual Lebesgue measure L. If $\frac{N\mu}{N-p} \le s \le \mu + p$, by the Hardy-Sobolev inequality (1.6), we also have that

$$\sup_n \int_{\Omega} |u_n|^{p^*(s,\mu)} d\nu < \infty. \tag{2.2}$$

Then by the Hölder inequality related to the measure ν, for any subset $\Lambda \subset \Omega$, since $1 \le q < p^*(s,\mu)$, we have

$$\begin{aligned} &\int_{\Lambda} |u_n - u|^q d\nu \\ &\le \Big(\int_{\Lambda} |u_n - u|^{p^*(s,\mu)} d\nu\Big)^{\frac{q}{p^*(s,\mu)}} \Big(\nu(\Lambda)\Big)^{\frac{p^*(s,\mu)-q}{p^*(s,\mu)}} \\ &\le C\Big(\nu(\Lambda)\Big)^{\frac{p^*(s,\mu)-q}{p^*(s,\mu)}}. \end{aligned} \tag{2.3}$$

Recalling that $\nu\big|_{\Omega}$ is absolutely continuous, we have $\nu(\Lambda) \to 0$ as $L(\Lambda) \to 0$. Hence,

$$\int_{\Lambda} \frac{|u_n - u|^q}{|x|^s} dx \to 0 \text{ as the measure } L(\Lambda) \to 0 \text{ uniformly for all } n. \tag{2.4}$$

Since Ω is bounded and $u_n \to u$ a.e. in Ω, applying the Egoroff Theorem and the above conclusion (2.4), we see that, up to a subsequence,

$$\int_{\Omega} \frac{|u_n - u|^q}{|x|^s} dx \to 0 \text{ as } n \to \infty.$$

When $s < \frac{N\mu}{N-p}, 1 \le q < p^*$, we denote $s_0 := \frac{N\mu}{N-p}$. Note that $p^*(s_0, \mu) = \frac{Np}{N-p} = p^*$. By the above arguments, if $1 \le q < p^*$, we have

$$u_n \to u \text{ strongly in } L^q(\Omega, \frac{dx}{|x|^{s_0}}). \tag{2.5}$$

Hence,

$$\int_\Omega \frac{|u_n - u|^q}{|x|^s} dx = \int_\Omega \frac{|u_n - u|^q}{|x|^{s_0}} |x|^{s_0 - s} dx \le C(\Omega) \int_\Omega \frac{|u_n - u|^q}{|x|^{s_0}} dx \to 0 \tag{2.6}$$

as $n \to \infty$. □

To consider the case with unbounded Ω, we insert the definition of tightness which can be found in [19, 20].

Definition 2.1. *Assume $\{\rho_k\}$ is a bounded sequence in $L^1(\mathbb{R}^N)$ and $\rho_k \ge 0$ satisfies*

$$\|\rho_k\|_{L^1} = \lambda + o(1), \ \lambda > 0. \tag{2.7}$$

Then we call this sequence $\{\rho_k\}$ is a tight sequence if $\forall\ \varepsilon > 0, \exists\ R > 0$ such that

$$\int_{|x| \ge R} \rho_k(x) dx < \varepsilon, \ \forall\ k \ge 1. \tag{2.8}$$

We call u_k is a L^p tight sequence, if $|u_k|^p$ is a tight sequence. For the convenience, the definition is still valid in the current paper when $\lambda = 0$ in (2.7).

Completion of the proof of Theorem 2.1. We assume that $s \ge \max\{0, \frac{N\mu}{N-p}\}$ and $L(\Omega) < \infty$. Note that in this case, $p^*(s, \mu) \le p^*$. Based on the results above, we only need to show that $\{\dfrac{|u_n|^q}{|x|^s}\}$ is a tight sequence. It is still satisfying that $\nu(\Omega) < \infty$ and $\nu\big|_\Omega$ is absolutely continuous, where $\nu = \frac{dx}{|x|^s}$. Hence,

$$L(\Omega \cap B_R^c(0)) \to 0 \text{ as } R \to \infty.$$

and then

$$\nu(\Omega \cap B_R^c(0)) \to 0 \text{ as } R \to \infty. \tag{2.9}$$

Apply the Hölder inequality on the domain $\Omega \cap B_R^c(0)$, it follows from (2.2) and (2.9) that $\{\dfrac{|u_n|^q}{|x|^s}\}$ is a tight sequence. □

3 The existence of extremal functions for a family of CKN inequalities

Firstly, consider the following problem:

$$-\,div\big(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{\mu}}\big)+\lambda\frac{|u|^{q-2}u}{|x|^{\sigma}}=\frac{|u|^{r-2}u}{|x|^{s}}\ \text{ in }\ \Omega,\ \lambda>0,\ u\in W^{1,p}_{\mu}(\Omega),\quad (3.1)$$

where $p>1,\max\{\sigma,s\}<\mu+p<N, q\geq 1, r\geq 1$ and Ω is an open Lipschitz domain of $\mathbb{R}^N$. We introduce the Sobolev space

$$E:=W^{1,p}_{\mu}(\Omega)\cap L^{q}(\Omega,\frac{dx}{|x|^{\sigma}}) \quad (3.2)$$

which is equipped with the norm

$$\|u\|_E:=\|u\|+\lambda|u|_{q,\sigma}, \quad (3.3)$$

where $\|u\|$ is the norm in $W^{1,p}_{\mu}(\Omega)$ defined by (1.14). Then we see that E is a closed subspace of $W^{1,p}_{\mu}(\Omega)$.

Lemma 3.1. *Assume that* $p>1,\max\{\sigma,s\}<\mu+p<N$, $1\leq q<\min\{p^*,p^*(\sigma,\mu)\}$, $\max\{\frac{p(\sigma-s)}{N-\mu-p}+q,\frac{\sigma-s}{N-\sigma}q+q\}<r<\min\{p^*,p^*(s,\mu)\}$ *and that*

$$\begin{cases}p(s-\sigma)+q(\mu+p-s)\leq r(\mu+p-\sigma),\\ (Np-Nr+pr)(s-\sigma)\geq (N\mu-Ns+ps)(r-q),\end{cases}$$

then there exists some constant $C(p,q,r,\mu,\sigma,s)>0$ *such that the CKN inequality* (1.10) *holds true, i.e.,*

$$\Big(\int_{\Omega}\frac{|u|^r}{|x|^s}dx\Big)^{\frac{1}{r}}\leq C(p,q,r,\mu,\sigma,s)\Big(\int_{\Omega}\frac{|\nabla u|^p}{|x|^{\mu}}dx\Big)^{\frac{a}{p}}\Big(\int_{\Omega}\frac{|u|^q}{|x|^{\sigma}}\Big)^{\frac{1-a}{q}} \quad (3.4)$$

for all $u\in E:=W^{1,p}_{\mu}(\Omega)\cap L^q(\Omega,\frac{dx}{|x|^{\sigma}})$*, where*

$$a=\frac{[(N-\sigma)r-(N-s)q]p}{[(N-\sigma)p-(N-\mu-p)q]r}\in(0,1).$$

Moreover, if $r\geq 1$, $E\hookrightarrow L^r(\Omega,\frac{dx}{|x|^s})$ *is a continuous embedding.*

Proof. Let

$$\begin{cases}r_1:=\frac{\big[p(N-s)-r(N-\mu-p)\big]q}{p(N-\sigma)-q(N-\mu-p)},\\ r_2:=\frac{\big[r(N-\sigma)-q(N-s)\big]p}{p(N-\sigma)-q(N-\mu-p)},\\ s_1:=\frac{\big[p(N-s)-r(N-\mu-p)\big]\sigma}{p(N-\sigma)-q(N-\mu-p)},\\ s_2:=\frac{Np(s-\sigma)+(r\sigma-qs)(N-\mu-p)}{p(N-\sigma)-q(N-\mu-p)},\\ \bar{\sigma}:=\frac{Np(s-\sigma)+(r\sigma-qs)(N-\mu-p)}{p(s-\sigma)+(r-q)(N-\mu-p)},\end{cases} \quad (3.5)$$

then a direct calculation shows that

$$\begin{cases} 0 < r_1 < q, r_2 > 0, r_1 + r_2 = r, \\ s_1 + s_2 = s, \frac{N\mu}{N-p} \le \bar{\sigma} \le \mu + p < N, \\ \frac{qs_1}{r_1} = \sigma, \\ \frac{qs_2}{q-r_1} = \bar{\sigma}, \\ \frac{qr_2}{q-r_1} = p^*(\bar{\sigma}, \mu) := \frac{p(N-\bar{\sigma})}{N-\mu-p} \in [p, p^*]. \end{cases} \tag{3.6}$$

Thus, by the Hölder inequality and Hardy-Sobolev inequality, we have

$$\begin{aligned} &\int_\Omega \frac{|u|^r}{|x|^s} dx = \int_\Omega \frac{|u|^{r_1}}{|x|^{s_1}} \cdot \frac{|u|^{r_2}}{|x|^{s_2}} dx \\ &\le \Big(\int_\Omega \frac{|u|^q}{|x|^\sigma} dx \Big)^{\frac{r_1}{q}} \Big(\int_\Omega \frac{|u|^{p^*(\bar{\sigma},\mu)}}{|x|^{\bar{\sigma}}} dx \Big)^{\frac{q-r_1}{q}} \\ &\le C(\bar{\sigma}) \Big(\int_\Omega \frac{|u|^q}{|x|^\sigma} dx \Big)^{\frac{r_1}{q}} \Big(\int_\Omega \frac{|\nabla u|^p}{|x|^\mu} dx \Big)^{\frac{p^*(\bar{\sigma},\mu)}{p} \frac{q-r_1}{q}}. \end{aligned} \tag{3.7}$$

Note that $0 < 1 - a = \frac{r_1}{r} < 1$ since $0 < r_1 < r$, we also have $\dfrac{p^*(\bar{\sigma}, \mu)(q - r_1)}{qr} = a$. Hence, we obtain that there exists some $C(p, q, r, \mu, \sigma, s) > 0$ such that (3.4) is satisfied for all $u \in W^{1,p}_\mu(\Omega) \cap L^q(\Omega, \frac{dx}{|x|^\sigma})$. Finally, if $r \ge 1$, by the Young inequality, we have

$$|u|_{r,s} \le \max\Big\{ \frac{1}{(1-a)\lambda}, \frac{1}{a} \Big\} C(p, q, r, \mu, \sigma, s) \|u\|_E, \tag{3.8}$$

where $\|u\|_E$ is defined by (3.3). Thus, $E \hookrightarrow L^r(\Omega, \frac{dx}{|x|^s})$ is a continuous embedding. □

Lemma 3.2. *Under the assumptions of Lemma 3.1, if furthermore $s > 0$ and the condition* (1.11) *strictly holds, i.e.,*

$$\begin{cases} p(s-\sigma) + q(\mu + p - s) < r(\mu + p - \sigma) \\ (Np - Nr + pr)(s - \sigma) > (N\mu - Ns + ps)(r - q) \end{cases}, \tag{3.9}$$

then any bounded sequence $\{u_n\}$ of $E := W^{1,p}_\mu(\Omega) \cap L^q(\Omega, \frac{dx}{|x|^\sigma})$ satisfying that $\{\frac{|u_n|^r}{|x|^s}\}$ is a tight sequence. In particular, if $r \ge 1$, the embedding

$$E \hookrightarrow L^r(\Omega, \frac{dx}{|x|^s})$$

is compact.

Proof. Let $\{u_n\} \subset E$ be a bounded sequence. Since $s > 0$, by the continuity, we can take some $0 < \bar{s} < s$ close to s such that the assumptions of Lemma 3.1 still hold after

replacing s by $\bar{s}$. Thus $\sup_n |u_n|_{r,\bar{s}} \le C$. Then it follows that

$$\begin{aligned} \int_{|x|>R} \frac{|u_n|^r}{|x|^s} dx &= \int_{|x|>R} \frac{1}{|x|^{s-\bar{s}}} \frac{|u_n|^r}{|x|^{\bar{s}}} dx \\ &< R^{\bar{s}-s} |u_n|^r_{r,\bar{s}} \\ &\to 0 \text{ uniformly for all } n \text{ as } R \to \infty. \end{aligned} \tag{3.10}$$

Hence, $\{\frac{|u_n|^r}{|x|^s}\}$ is a tight sequence. Recalling the Theorem 2.1 for the case bounded domain, it is easy to prove that the embedding $E \hookrightarrow L^r(\mathbb{R}^N, \frac{dx}{|x|^s})$ is compact. $\square$

Corollary 3.1. *Let $\sigma = 0, 1 < p = q < N$ and one of the following holds:*

(i) $\mu = 0, 0 < s < p < N, p \le r < \frac{p(N-s)}{N-p}$;

(ii) $\mu > 0, \frac{N\mu(r-p)}{p^2} < s < \mu + p < N, p \le r < \min\{\frac{pN}{N-p}, \frac{p(N-s)}{N-\mu-p}\}$;

(iii) $\mu < 0, 0 < s < \mu + p < N, p \le r < \frac{p(N-s)}{N-\mu-p}$,

then

$$W^{1,p}_\mu(\Omega) \cap L^p(\Omega) \hookrightarrow L^r(\Omega, \frac{dx}{|x|^s})$$

is a compact embedding.

Proof. We note that for case $r = p$, we can apply the similar arguments as the proof of Lemma 3.2. And other cases are straight-forward results of Lemma 3.2. $\square$

Remark 3.1. *It is well known that $H^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is a continuous embedding for $r \in [p, p^*)$ but not compact. Take $\mu = 0, 1 < p < N, 0 < s < p$, then by (i) of Corollary 3.1 we see that $H^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N, \frac{dx}{|x|^s})$ is a compact embedding for $r \in [p, p^*(s))$.*

Lemma 3.3. *Assume that $a > 0, b > 0, A > 0, B > 0$ are fixed, let $g(t) := t^a A + t^{-b} B$, then*

$$\inf_{t>0} g(t) = g(t_0) = \frac{a+b}{a} (\frac{b}{a})^{\frac{-b}{a+b}} A^{\frac{b}{a+b}} B^{\frac{a}{a+b}},$$

where $t_0 = (\frac{bB}{aA})^{\frac{1}{a+b}}$.

Proof. It is a direct computation. $\square$

Theorem 3.1. *Let Ω be a cone, that is, $\Omega = \frac{\Omega}{t}$ for any $t > 0$ (in particular, $\Omega = \mathbb{R}^N$). Assume that $p > 1, \max\{\sigma, s\} < \mu + p < N$, $1 \le q < \min\{p^*, p^*(\sigma, \mu)\}$, $\max\{\frac{p(\sigma - s)}{N - \mu - p} + q, \frac{\sigma - s}{N - \sigma} q + q\} < r < \min\{p^*, p^*(s, \mu)\}$ and*

$$\begin{cases} p(s-\sigma) + q(\mu + p - s) \le r(\mu + p - \sigma) \\ (Np - Nr + pr)(s - \sigma) \ge (N\mu - Ns + ps)(r - q) \end{cases},$$

then

$$
\begin{aligned}
&\inf_{u\in M} I(u)\\
=&\frac{(\mu+p-\sigma)r+(p-q)(N-s)}{p(N-s)-(N-\mu-p)r}\Big[\frac{(N-\sigma)r-(N-s)q}{p(N-s)-(N-\mu-p)r}\Big]^{\frac{(N-s)q-(N-\sigma)r}{(p-q)(N-s)+(\mu+p-\sigma)r}}\\
&\lambda^{\frac{(N-s)p-(N-\mu-p)r}{(\mu+p-\sigma)r+(p-q)(N-s)}}\inf_{u\in M}\Big\{\|u\|^{\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}|u|_{q,\sigma}^{\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}}\Big\},
\end{aligned}
\tag{3.11}
$$

where

$$
M:=\{u\in E:\int_\Omega\frac{|u|^r}{|x|^s}dx=1\},\tag{3.12}
$$

and

$$
I(u):=\int_\Omega\frac{|\nabla u|^p}{|x|^\mu}dx+\lambda\int_\Omega\frac{|u|^q}{|x|^\sigma}dx.\tag{3.13}
$$

Proof. For $t>0$, define a mapping $T_t(u)=u_t:=t^{\frac{N-s}{r}}u(tx)$, then it is easy to check that

$$
\int_\Omega\frac{|u|^r}{|x|^s}dx\equiv\int_\Omega\frac{|u_t|^r}{|x|^s}dx, t>0.
$$

Hence, M is invariant under the transformation T_t. We note that

$$
\frac{p(N-s)-(N-\mu-p)r}{r}>0,\frac{(N-\sigma)r-(N-s)q}{r}>0.
$$

Hence, by Lemma 3.3, we have

$$
\begin{aligned}
&\inf_{t>0} I(u_t)\\
=&\inf_{t>0}\Big(t^{\frac{p(N-s)-(N-\mu-p)r}{r}}\int_\Omega\frac{|\nabla u(x)|^p}{|x|^\mu}dx+t^{-\frac{r(N-\sigma)-q(N-s)}{r}}\lambda\int_\Omega\frac{|u(x)|^q}{|x|^\sigma}dx\Big)\\
=&\frac{(\mu+p-\sigma)r+(p-q)(N-s)}{p(N-s)-(N-\mu-p)r}\Big[\frac{(N-\sigma)r-(N-s)q}{p(N-s)-(N-\mu-p)r}\Big]^{\frac{(N-s)q-(N-\sigma)r}{(p-q)(N-s)+(\mu+p-\sigma)r}}\\
&\lambda^{\frac{(N-s)p-(N-\mu-p)r}{(\mu+p-\sigma)r+(p-q)(N-s)}}\|u\|^{\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}|u|_{q,\sigma}^{\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}}\\
=:&C^*(p,q,r,\mu,\sigma,s,\lambda)\|u\|^{\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}|u|_{q,\sigma}^{\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}}.
\end{aligned}
$$

Thereby, we prove this Lemma. □

Remark 3.2. *Define*

$$
\begin{aligned}
\lambda^*(p,q,r,\mu,\sigma,s):=&\Big\{\frac{p(N-s)-(N-\mu-p)r}{(\mu+p-\sigma)r+(p-q)(N-s)}\Big\}^{\frac{(\mu+p-\sigma)r+(p-q)(N-s)}{(N-s)p-(N-\mu-p)r}}\\
&\cdot\Big\{\frac{(N-\sigma)r-(N-s)q}{p(N-s)-(N-\mu-p)r}\Big\}^{\frac{(N-\sigma)r-(N-s)q}{p(N-s)-(N-\mu-p)r}},
\end{aligned}
\tag{3.14}
$$

then we have

$$C^*\big(p,q,r,\mu,\sigma,s,\lambda^*(p,q,r,\mu,\sigma,s)\big)\equiv 1.$$

For the simplicity, if there exists no misunderstanding, we will write

$$\lambda^*=\lambda^*(p,q,r,\mu,\sigma,s);\quad I^*(u)=\int_\Omega\frac{|\nabla u|^p}{|x|^\mu}dx+\lambda^*\int_\Omega\frac{|u|^q}{|x|^\sigma}dx.$$

Corollary 3.2. *Let Ω be a cone (i.e., $\Omega=\frac{\Omega}{t}$ for any $t>0$. In particular, $\Omega=\mathbb{R}^N$). Assume that $p>1,\max\{\sigma,s\}<\mu+p<N$, $1\le q<\min\{p^*,p^*(\sigma,\mu)\}$, $\max\{\dfrac{p(\sigma-s)}{N-\mu-p}+q,\dfrac{\sigma-s}{N-\sigma}q+q\}<r<\min\{p^*,p^*(s,\mu)\}$ and*

$$\begin{cases}p(s-\sigma)+q(\mu+p-s)\le r(\mu+p-\sigma)\\(Np-Nr+pr)(s-\sigma)\ge(N\mu-Ns+ps)(r-q)\end{cases},$$

then the sharp constant of (3.4)

$$C(p,q,r,\mu,\sigma,s)=\Big(\frac{1}{\rho}\Big)^{\frac{(\mu+p-\sigma)q+(p-r)(N-s)}{q[(N-\sigma)p-(N-\mu-p)r]}},$$

where $\rho:=\inf_{u\in M}I^*(u)$.

Proof. For any $u\in M$, by Theorem 3.1, we have

$$\inf_{t>0}I^*(u_t)=\|u\|^{\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}|u|_{q,\sigma}^{\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}}. \tag{3.15}$$

It follows that

$$\rho:=\inf_{u\in M}I^*(u)=\inf_{u\in M}\|u\|^{\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}|u|_{q,\sigma}^{\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}}. \tag{3.16}$$

Note that

$$\begin{aligned}&\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}+\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}\\&=\frac{r[(N-\sigma)p-(N-\mu-p)q]}{(\mu+p-\sigma)r+(p-q)(N-s)},\end{aligned}$$

then we have that

$$\rho|u|_{r,s}^{\frac{r[(N-\sigma)p-(N-\mu-p)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}\le\|u\|^{\frac{p[(N-\sigma)r-(N-s)q]}{(\mu+p-\sigma)r+(p-q)(N-s)}}|u|_{q,\sigma}^{\frac{q[(N-s)p-(N-\mu-p)r]}{(\mu+p-\sigma)r+(p-q)(N-s)}}$$

for all $u\in E$, it follows that

$$|u|_{r,s}\le\Big(\frac{1}{\rho}\Big)^{\frac{(\mu+p-\sigma)r+(p-q)(N-s)}{r[(N-\sigma)p-(N-\mu-p)q]}}\|u\|^a|u|_{q,\sigma}^{1-a}\ \text{for all } u\in E. \tag{3.17}$$

Note that the above processes are reversible, the Corollary is proved. $\square$

Remark 3.3. *Under the assumptions of Corollary 3.2, the sharp constant of inequality (3.4) can be achieved if and only if ρ can be reached.*

Next, let us assume $r \geq 1$ and consider the following minimizing problem.

Lemma 3.4. *Assume that $p > 1, \max\{\sigma, s\} < \mu + p < N$, $r \geq 1, 1 \leq q < \min\{p^*, p^*(\sigma, \mu)\}, \max\{\frac{p(\sigma - s)}{N - \mu - p} + q, \frac{\sigma - s}{N - \sigma} q + q\} < r < \min\{p^*, p^*(s, \mu)\}, s > 0$ and*

$$\begin{cases} p(s - \sigma) + q(\mu + p - s) < r(\mu + p - \sigma) \\ (Np - Nr + pr)(s - \sigma) > (N\mu - Ns + ps)(r - q) \end{cases},$$

then ρ can be achieved by some minimizer $u \in M$. Furthermore, if $q > 1, r > 1$, the minimizer is a ground state solution to the following problem:

$$-div(\frac{|\nabla u|^{p-2}\nabla u}{|x|^\mu}) + \frac{q\lambda^*}{p} \frac{|u|^{q-2}u}{|x|^\sigma} = \frac{r[p(N-\sigma) - (N - \mu - p)q]\rho}{p[(\mu + p - \sigma)r + (p - q)(N - s)]} \frac{|u|^{r-2}u}{|x|^s}, \tag{3.18}$$

for $u \in E$, where ρ is defined in Corollary 3.2.

Proof. Obviously, $\lambda^* > 0$. Let $\{u_n\}$ be a minimizing sequence of ρ in M, i.e., $|u_n|_{r,s} \equiv 1$ and $I^*(u_n) \to \rho$. By (3.8), $\rho > 0$. Further, $\{u_n\}$ is bounded in E. By Lemma 3.2, up to a subsequence if necessary, we may assume that $u_n \to u$ in $L^r(\Omega, \frac{dx}{|x|^s})$. Hence, $u \in M$. We also note that $\sup_{n \geq 1} \|u_n\| < \infty$, then by Lemma 2.1, we may assume that $u_n \to u$ a.e. in Ω since $p > 1$. Follows from the Fatou's Lemma, we have $I^*(u) \leq \liminf_{n\to\infty} I^*(u_n) = \rho$. On the other hand, by the definition of ρ, we have $I^*(u) \geq \rho$ since $u \in M$. Hence, u is a minimizer. Let u be an extremal function. If $q > 1, r > 1$, then there exists some Lagrange multiplier $\tilde{\lambda}$ such that

$$- p\ div(\frac{|\nabla u|^{p-2}\nabla u}{|x|^\mu}) + q\lambda^* \frac{|u|^{q-2}u}{|x|^\sigma} = \tilde{\lambda} \frac{|u|^{r-2}u}{|x|^s}. \tag{3.19}$$

Testing by u, we obtain that

$$p\|u\|^p + q\lambda^*|u|^q_{q,\sigma} = \tilde{\lambda}. \tag{3.20}$$

Recalling that

$$\|u\|^p + \lambda^*|u|^q_{q,\sigma} = \rho \tag{3.21}$$

and by Lemma 3.3, we see that

$$\frac{p(N-s) - (N - \mu - p)r}{r} \|u\|^p = \frac{r(N-\sigma) - q(N-s)}{r} \lambda^*|u|^q_{q,\sigma}. \tag{3.22}$$

Combine (3.20), (3.21) and (3.22), we obtain that

$$\tilde{\lambda} = \frac{r[p(N-\sigma) - (N - \mu - p)q]}{(\mu + p - \sigma)r + (p - q)(N - s)} \rho. \tag{3.23}$$

Hence, the minimizer is a ground state solution to the equation (3.18). $\square$

Proof of Theorem 1.1. It is a straightforward consequence of Corollary 3.2 and Lemma 3.4. $\square$

4 An application

In this section, we will study the problem (1.15) as an application of the previous theorem. Based on the Hardy-Sobolev inequality (1.6), firstly, we get the following weakly continuous functional, which largely generalizes the corresponding result in [27, Lemma 2.13].

Lemma 4.1. *If* $1 < p < N, \mu + p < N, q < p^* := \frac{pN}{N-p}$ *and there exists some* $\frac{N\mu}{N-p} \leq \eta < \min\{\mu + p, N - \frac{q}{p}(N - \mu - p)\}$ *such that* $\left|a(x)\right|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}} |x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} \in L^1(\Omega)$, *then the functional* $\chi : W_\mu^{1,p}(\Omega) \to \mathbb{R}$ *defined by*

$$\chi(u) = \int_\Omega a(x)|u|^q dx$$

is weakly continuous. In particular, when $\mu \leq 0 < \mu + p < N, q = p$ *and* $a(x) \in L^{\frac{N}{\mu+p}}(\Omega)$, *the result holds.*

Proof. By $\frac{N\mu}{N-p} \leq \eta < \min\{\mu + p, N - \frac{q}{p}(N - \mu - p)\}$, we have $p^* \geq p^*(\eta, \mu) > \max\{p, q\}$. Then by the Hölder inequality, for any $u \in W_\mu^{1,p}(\Omega)$, we have

$$\begin{aligned} &\int_\Omega |a(x)||u|^q dx = \int_\Omega \left|a(x)|x|^{\frac{q\eta}{p^*(\eta,\mu)}}\right| \frac{|u|^q}{|x|^{\frac{q\eta}{p^*(\eta,\mu)}}} dx \\ &\leq \Big(\int_\Omega \left|a(x)\right|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}} |x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} dx\Big)^{\frac{p^*(\eta,\mu)-q}{p^*(\eta,\mu)}} \Big(\int_\Omega \frac{|u|^{p^*(\eta,\mu)}}{|x|^\eta} dx\Big)^{\frac{q}{p^*(\eta,\mu)}}. \end{aligned}$$

By the Hardy-Sobolev inequality (1.6) and the assumption that $\left|a(x)\right|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}} |x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} \in L^1(\Omega)$, we see that $\chi(u)$ is well defined. Now we assume that $u_n \rightharpoonup u$ in $W_\mu^{1,p}(\Omega)$. By Lemma 2.1, going to a subsequence if necessary, we may assume that

$$u_n \to u \ a.e. \text{ on } \Omega.$$

By the Hardy-Sobolev inequality again, we see that $\{u_n\}$ is bounded in $L^{p^*(\eta,\mu)}(\Omega, \frac{dx}{|x|^\eta})$, then $\{\frac{|u_n|^q}{|x|^{\frac{q\eta}{p^*(\eta,\mu)}}}\}$ is bounded in $L^{\frac{p^*(\eta,\mu)}{q}}(\Omega)$. Hence $\frac{|u_n|^q}{|x|^{\frac{q\eta}{p^*(\eta,\mu)}}} \rightharpoonup \frac{|u|^q}{|x|^{\frac{q\eta}{p^*(\eta,\mu)}}}$ in $L^{\frac{p^*(\eta,\mu)}{q}}(\Omega)$ up to a subsequence. Recalling that $a(x)|x|^{\frac{q\eta}{p^*(\eta,\mu)}} \in L^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}}$ and

$$\frac{1}{\frac{p^*(\eta,\mu)}{q}} + \frac{1}{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}} = 1,$$

we obtain that

$$\int_\Omega a(x)|u_n|^p dx \to \int_\Omega a(x)|u|^p dx.$$

Thus, we prove that χ is weakly continuous. Especially, when $\mu \leq 0 < \mu + p < N$ and $q = p$, we can take $\eta = 0$ and obtain the final result. $\square$

Remark 4.1. *When $q = p = 2, \mu = 0$, Lemma 4.1 is exactly the Lemma 2.13 in [27]. Evidently, such a very typical case is essentially different from the general situation considered here.*

Consider the minimizing problem

$$(Q)\ \min\Big\{\int_\Omega \frac{|\nabla u|^p}{|x|^\mu}dx:\ u\in W^{1,p}_\mu(\Omega), \int_\Omega V|u|^q dx = 1\Big\}.$$

Lemma 4.2. *Under the assumption* $(\mathbf{H})$, $\int_\Omega V_+|u|^q dx$ *is weakly continuous in* $W^{1,p}_\mu(\Omega)$.

Proof. The proof is inspired by that of [23, Lemma 2.1]. However, our case is much more complicated. In view of Lemma 4.1, we only need to prove that $\int_\Omega V_2|u|^p dx$ is weakly continuous.

Step 1. We prove that $\{V_2|u_n|^q\}$ is a tight sequence. We only need to prove the case of (H_2) or (H_3). For the case of (H_2), by the Hölder inequality and the Hardy-Sobolev inequality (1.6), we see that

$$\begin{aligned}
&\int_{\Omega\cap B^c_R(0)} V_2|u_n|^q dx = \int_{\Omega\cap B^c_R(0)} V_2|x|^{\frac{(\mu+p)q}{p}}\frac{|u_n|^q}{|x|^{\frac{(\mu+p)q}{p}}}dx\\
&\le \Big(\int_{\Omega\cap B^c_R(0)}\big(V_2(x)\big)^{\frac{p}{p-q}}|x|^{\frac{(\mu+p)q}{p-q}}dx\Big)^{\frac{p-q}{p}}\Big(\int_{\Omega\cap B^c_R(0)}\frac{|u_n|^p}{|x|^{\mu+p}}dx\Big)^{\frac{q}{p}}\\
&\to 0 \text{ as } R\to\infty.
\end{aligned}$$

For the case of (H_3), since $q \ge p, \mu+p < N$, we have $\bar\sigma := N - \frac{q}{p}(N-\mu-p) \le \mu+p$ and $q = p^*(\bar\sigma,\mu)$. Recall that $\lim\limits_{\substack{|x|\to\infty\\ x\in\Omega}} |x|^{\bar\sigma}V_2(x) = 0$ and the Hardy-Sobolev inequality, we see that $\{V_2|u_n|^q\}$ is also a tight sequence. In summary, under the assumption (H), for $\forall\ \varepsilon > 0$, we can take $R > 0$ large enough such that

$$\int_{\Omega\setminus B_R(0)} V_2|u_n|^q dx < \varepsilon \text{ for all } n. \tag{4.1}$$

It follows from Fatou's Lemma, we also have

$$\int_{\Omega\setminus B_R(0)} V_2|u|^q dx < \varepsilon. \tag{4.2}$$

Step 2. We note that for the cases of (H_1) and (H_2), $1 < q < p$. Due to the compactness, we can choose a finite covering of $\overline{\Omega\cap B_R(0)}$ by closed balls $\overline{B_{r_i}(x_i)}, 1\le i\le k$ such that $\{x_i\}_{i=1}^k \subset \overline{\Omega\cap B_R(0)} \subset \bar\Omega$ and that

$$|x-x_i|^{\mu+p}V_2(x) \le C \text{ for all } x\in B_{r_i}(x_i), i = 1,2,\cdots,k. \tag{4.3}$$

Note that $1 < q < p = p^*(\mu+p,\mu)$, then by the weighted Rellich-Kondrachov compactness Theorem 2.1, it is easy to obtain that

$$\int_{\Omega\cap B_R(0)} V_2|u_n|^q dx \to \int_{\Omega\cap B_R(0)} V_2|u|^q dx \text{ as } n\to\infty. \tag{4.4}$$

Hence, for the cases of (H_1) and (H_2), by (4.1), (4.3) and (4.4), we prove that

$$\int_\Omega V_2|u_n|^q dx \to \int_\Omega V_2|u|^q dx \text{ as } n \to \infty.$$

For the case of (H_3), since $q \geq p, \mu + p < N$, we see that $p^*(\bar{s}, \mu) \geq \max\{p, q\}$. By compactness again, for $\forall\ \varepsilon > 0$, we can choose a finite covering of $\overline{\Omega \cap B_R(0)}$ by closed balls $\overline{B_{r_i}(x_i)}, 1 \leq i \leq k$ such that $\{x_i\}_{i=1}^k \subset \overline{\Omega \cap B_R(0)} \subset \bar{\Omega}$ and

$$|x - x_i|^{\bar{s}} V_2(x) \leq \varepsilon \text{ for all } x \in B_{r_i}(x_i), \ \ i = 1, 2, \cdots, k. \tag{4.5}$$

We note that k depends on ε. By the assumption (H_3) again, we can take $0 < r < \min\{r_1, r_2, \cdots, r_k\}$ such that

$$|x - x_i|^{\bar{s}} V_2(x) \leq \frac{\varepsilon}{k} \text{ for all } x \in B_r(x_i), \ \ i = 1, 2, \cdots, k. \tag{4.6}$$

Set

$$A := \bigcup_{i=1}^k B_r(x_i),$$

then by the Hardy-Sobolev inequality (1.6),

$$\Big(\int_{B_r(x_i)} \frac{|u_n|^q}{|x - x_i|^{\bar{s}}} dx\Big)^{\frac{1}{q}} \leq C\Big(\int_{B_r(x_i)} \frac{|\nabla u_n|^p}{|x - x_i|^\mu} dx\Big)^{\frac{1}{p}}, \ i = 1, 2, \cdots, k. \tag{4.7}$$

Thus,

$$\int_A V_2|u_n|^q dx \leq \varepsilon C^q, \int_A V_2|u|^q dx \leq \varepsilon C^q. \tag{4.8}$$

It follows from (4.5) that $V_2 \in L^\infty\big((\Omega \cap B_R(0))\backslash A\big)$, and then V_2 satisfies the assumption of Lemma 4.1 up to the bounded domain $(\Omega \cap B_R(0))\backslash A$. So

$$\int_{(\Omega \cap B_R(0))\backslash A} V_2|u_n|^q dx \to \int_{(\Omega \cap B_R(0))\backslash A} V_2|u|^q dx. \tag{4.9}$$

Then, by (4.1), (4.2), (4.8) and (4.9), we also have

$$\int_\Omega V_2|u_n|^q dx \to \int_\Omega V_2|u|^q dx.$$

□

Corollary 4.1. *Under the assumption (H), $\int_\Omega V(x)|u|^q dx$ is weakly upper semicontinuous in $W_\mu^{1,p}(\Omega)$.*

Proof. It is an obvious conclusion which can be deduced by Lemma 4.2 and the Fatou's Lemma. □

Theorem 4.1. *Under the assumption (H), problem (Q) has a solution $\varphi_1 \geq 0$. Moreover, (φ_1, λ_1) is a solution to problem (1.15), where $\lambda_1 := \int_\Omega \frac{|\nabla \varphi_1|^p}{|x|^\mu} dx$.*

Proof. Let $\{u_n\}$ be a minimizing sequence for (Q). By Lemma 2.1, we may assume that $u_n \rightharpoonup u$ in $W^{1,p}_{\mu}(\Omega)$ and $u_n \to u$ a. e. on Ω. Hence,

$$\int_{\Omega} \frac{|\nabla u|^p}{|x|^{\mu}} dx \leq \liminf_{n\to\infty} \int_{\Omega} \frac{|\nabla u_n|^p}{|x|^{\mu}} dx = \inf(Q).$$

By Corollary 4.1, we have that $\displaystyle\int_{\Omega} V|u|^q dx \geq 1$. Let

$$\varphi_1 := \frac{u}{(\int_{\Omega} V|u|^q dx)^{\frac{1}{q}}},$$

we see that $\int_{\Omega} V|\varphi_1|^q dx = 1$ and

$$\inf(Q) \leq \int_{\Omega} \frac{|\nabla \varphi_1|^p}{|x|^{\mu}} dx = \frac{1}{(\int_{\Omega} V|u|^q dx)^{\frac{p}{q}}} \int_{\Omega} \frac{|\nabla u|^p}{|x|^{\mu}} dx \leq \int_{\Omega} \frac{|\nabla u|^p}{|x|^{\mu}} dx \leq \inf(Q).$$

Hence, we see that $\int_{\Omega} V|u|^q dx = 1$ and $\varphi_1 = u$ is a solution of (Q). Note that $|\varphi_1|$ is also a solution, we may assume $\varphi_1 \geq 0$. Moreover, there exists some Lagrange multiplier λ_1 such that

$$-div(\frac{|\nabla \varphi_1|^{p-2} \nabla \varphi_1}{|x|^{\mu}}) = \lambda_1 V(x) |\varphi_1|^{q-2} \varphi_1.$$

Testing by φ_1, we have

$$\int_{\Omega} \frac{|\nabla \varphi_1|^p}{|x|^{\mu}} dx = \lambda_1 \int_{\Omega} V(x) |\varphi_1|^q dx = \lambda_1 > 0.$$

We also note that $\lambda_1 = \inf(Q)$. ◻

We need the following Brézis-Lieb type lemma.

Lemma 4.3. *Let Ω be an open subset of $\mathbb{R}^N$ and assume that $\{u_n\}$ satisfies*

$$q \geq 1, \sup_n \int_{\Omega} |a(x)||u_n|^q dx < \infty \textit{ and } u_n \to u \; a.e. \textit{ in } \Omega.$$

Then

$$\lim_{n\to\infty} \int_{\Omega} a(x)(|u_n|^q - |u_n - u|^q) = \int_{\Omega} a(x)|u|^q dx. \tag{4.10}$$

Proof. Consider the new measure ν such that $d\nu = |a(x)|dx$, then (4.10) can be deduced from the Brézis-Lieb Lemma with respect to the new measure ν:

$$\lim_{n\to\infty} \int_{\Omega} \Big| |u_n|^q - |u_n - u|^q - |u|^q \Big| d\nu = 0.$$

◻

Now we introduce a new space $E := \{u \in W^{1,p}_\mu(\Omega) : \|u\|_E < \infty\}$, where

$$\|u\|_E := \Big(\int_\Omega \frac{|\nabla u|^p}{|x|^\mu}dx\Big)^{\frac{1}{p}} + \Big(\int_\Omega V_-|u|^q dx\Big)^{\frac{1}{q}}.$$

Set

$$I(u) := \int_\Omega \frac{|\nabla u|^p}{|x|^\mu}dx, \quad J(u) := \int_\Omega V|u|^q dx, \quad J_\pm(u) := \int_\Omega V_\pm|u|^q dx,$$

then we see that $M := \{u \in E : J(u) = 1\}$ is a C^1-manifold.

Lemma 4.4. *If V satisfies (H), then*

(i) *there exists some $C > 0$ such that $J_+(u) \le C\big(I(u)\big)^{\frac{q}{p}}$ for all $u \in E$;*

(ii) *J_+ is weakly continuous and J_+' is completely continuous (or weak-to-strong continuous), i.e., if $u_n \rightharpoonup u$, then $J_+'(u_n) \to J_+'(u)$.*

Proof. Let $u_n \rightharpoonup u$ in E, since $E \subset W^{1,p}_\mu(\Omega)$, we may assume that $u_n \to u$ a.e. in Ω. We note that (i) is deduced by Lemma 4.2. Next we shall prove (ii). By Lemma 4.3, we see that J_+ is weakly continuous. For any $v \in E$, by the Hölder inequality up to the new measure $d\nu = V_+dx$, we have

$$\begin{aligned}
&\Big|\big\langle J_+'(u_n) - J_+'(u), v\big\rangle\Big| \\
=&\Big|\int_\Omega V_+\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)vdx\Big| \\
=&\Big|\int_\Omega \big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)vd\nu\Big| \\
\le&\Big(\int_\Omega \big||u_n|^{q-2}u_n - |u|^{q-2}u\big|^{\frac{q}{q-1}}d\nu\Big)^{\frac{q-1}{q}}\Big(\int_\Omega |v|^q d\nu\Big)^{\frac{1}{q}} \\
\le&C\|v\|_E\Big(\int_\Omega \big||u_n|^{q-2}u_n - |u|^{q-2}u\big|^{\frac{q}{q-1}}d\nu\Big)^{\frac{q-1}{q}}.
\end{aligned} \tag{4.11}$$

Let $v_n := |u_n|^{q-2}u_n, v := |u|^{q-2}u$. Since $\frac{q}{q-1} > 1$, by Lemma 4.3 we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \big||u_n|^{q-2}u_n - |u|^{q-2}u\big|^{\frac{q}{q-1}}d\nu \\
&= \lim_{n\to\infty}\int_\Omega |v_n - v|^{\frac{q}{q-1}}d\nu \\
&= \lim_{n\to\infty}\int_\Omega \big(|v_n|^{\frac{q}{q-1}} - |v|^{\frac{q}{q-1}}\big)d\nu \\
&= \lim_{n\to\infty}\int_\Omega \big(|u_n|^q - |u|^q\big)d\nu \\
&= 0 \text{ since } J_+ \text{ is weakly continuous.}
\end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.12) that J_+' is completely continuous. $\square$

Proof of Theorem 1.2. For the convenience of the readers, we recall the Krasnoselskii Genus. Define $\mathcal{A} = \{A \subset M : A \text{ closed}, A = -A\}$. For $A \subset \mathcal{A}$, $A \neq \emptyset$, let

$$\gamma(A) := \begin{cases} \inf\{m : \ \exists\, h \in C^0(A; \mathbb{R}^m \backslash \{0\}), h(-u) = -h(u)\}, \\ \infty, \quad \text{if } \{\cdots\} = \emptyset, \text{ in particular, if } 0 \in A, \end{cases}$$

and let $\gamma(\emptyset) = 0$. Define

$$\lambda_n := \inf_{\gamma(A) \geq n} \sup_{u \in A} I(u), \quad n = 1, 2, \cdots.$$

Under the assumption (H), we see that $\{x \in \Omega : V(x) > 0\}$ has positive measure. Note that $\gamma(\mathcal{S}^{n-1}) = n$, where $\mathcal{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$, it follows that λ_n is well defined for all n by constructing a suitable odd homeomorphism. Moreover, we see that $\lambda_1 = \inf_{u \in M} I(u) > 0$ coincides with the value given by Theorem 4.1. We now prove that $I\Big|_M$ satisfies PS condition. Let $\{u_n\} \subset E$ be a PS sequence. Then there is a corresponding sequence $\mu_k \in \mathbb{R}$ such that

$$A_{\mu_k}(u_k) := I'(u_k) - \mu_k J'(u_k) \to 0 \text{ in } E^*. \tag{4.13}$$

It follows from (i) of Lemma 4.4 that $J_+(u_k)$ is bounded and therefore $J_-(u_k)$ is bounded since $J_-(u_k) = J_+(u_k) - 1$. Note that $\|u_k\|_E \equiv I(u_k)^{\frac{1}{p}} + J_-(u_k)^{\frac{1}{q}}$, we see that $\{u_k\}$ is bounded in E and then $J'(u_k)$ is bounded. Up to a subsequence, we may assume that $u_k \rightharpoonup u$ in E and $u_k \to u$ a.e. in Ω. By Corollary 4.1, we have $J(u) \geq 1$ and it follows that

$$\langle J'(u), u \rangle = qJ(u) \geq q. \tag{4.14}$$

Testing by u_k in (4.13), we get that

$$\langle I'(u_k), u_k \rangle - \mu_k \langle J'(u_k), u_k \rangle = pI(u_k) - q\mu_k \to 0. \tag{4.15}$$

By the boundedness of $I(u_k)$ and (4.15), we obtain that $\{\mu_k\}$ is bounded. Up to a subsequence, we assume that $\mu_k \to \mu_\infty$ and it follows from (4.15) again, we have $I(u_k) \to \frac{q}{p}\mu_\infty$ and $A_{\mu_\infty}(u_k) \to 0, \langle A_{\mu_\infty}(u_k), u_k - u \rangle \to 0$. By the Brézis-Lieb Lemma, it is easy to see that

$$I(u_k - u) = I(u_k) - I(u) + o(1). \tag{4.16}$$

Insert Lemma 4.3, we have

$$J_-(u_k - u) = J_-(u_k) - J_-(u) + o(1). \tag{4.17}$$

A direct calculation shows that

$$\langle A_{\mu_\infty}(u_k), u_k - u \rangle = p\Big[I(u_k) - I(u)\Big] + q\mu_\infty\Big[\big(J_-(u_k) - J_-(u)\big)\Big].$$

Combine with (4.16) and (4.17) that

$$pI(u_k - u) + q\mu_\infty J_-(u_k - u) \to 0. \tag{4.18}$$

By the weakly lower semicontinuity of a norm and (4.15), we have

$$I(u) \le \liminf_{k\to\infty} I(u_k) = \frac{q}{p}\mu_\infty.$$

Since $J(u) \ge 1$, we have $u \ne 0$ and then $I(u) > 0$. Thus, $\mu_\infty > 0$ and further by (4.18), we have $I(u_k - u) \to 0, J_-(u_k - u) \to 0$. Hence,

$$\|u_k - u\|_E \equiv I(u_k - u)^{\frac{1}{p}} + J_-(u_k - u)^{\frac{1}{q}} \to 0.$$

We have thus proved that $I\Big|_M$ satisfies PS condition. Hence, $\lambda_n s$ are critical values due to [22, page 98, Theorem 5.7]. There exists a critical point φ_n such that $I'(\varphi_n) = \widetilde{\mu} J'(\varphi_n)$. Testing by φ_n, we have

$$pI(\varphi_n) = \langle I'(\varphi_n), \varphi_n\rangle = \widetilde{\mu}\langle J'(\varphi_n), \varphi_n\rangle = q\widetilde{\mu}J(\varphi_n) = q\widetilde{\mu}.$$

Hence, $\lambda_n = \frac{q}{p}\widetilde{\mu} = I(\varphi_n)$. Finally, we shall prove that $\lambda_n \to \infty$. Note that if $\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+k-1} = \lambda$ for some n, k, then the set of critical points corresponding to λ has genus $\ge k$ (see [22, page 97, Lemma 5.6]). We also note that $\lambda_{n+1} \ge \lambda_n$, apply the similar argument of [21, Proposition 9.33], we can prove that $\lambda_n \to \infty$ as $n \to \infty$. Moreover, if $q = p$, we see that $\lambda_n s$ are eigenvalues such that $\lambda_n \to \infty$ as $n \to \infty$ and φ_n is an eigenfunction of (1.15) corresponding to λ_n. On the other hand, if $1 < q \ne p$, after scaling we see that $v_n := (\frac{\lambda}{\lambda_n})^{\frac{1}{p-q}}\varphi_n$ is a sequence of solutions to (1.15) such that

$$\begin{aligned}
c_n :=& \Phi(v_n) =: \frac{1}{p}I(v_n) - \frac{1}{q}\lambda J(v_n)\\
=&(\frac{1}{p} - \frac{1}{q})I(v_n)\\
=&(\frac{1}{p} - \frac{1}{q})(\frac{\lambda}{\lambda_n})^{\frac{p}{p-q}} I(u_n)\\
=&(\frac{1}{p} - \frac{1}{q})(\frac{\lambda}{\lambda_n})^{\frac{p}{p-q}} \lambda_n\\
=&(\frac{1}{p} - \frac{1}{q})\lambda^{\frac{p}{p-q}} \lambda_n^{\frac{q}{q-p}}.
\end{aligned} \tag{4.19}$$

Hence, if $q > p$, (1.15) possesses a sequence of solutions with energy $0 < c_1 \le c_2 \le \cdots \le c_n \to \infty$ as $n \to \infty$. If $q < p$, (1.15) has a sequence of solutions with energy $-\infty < c_1 \le c_2 \le \cdots \le c_n \to 0$ as $n \to \infty$. □

As an application, we obtain the following result:

Corollary 4.2. *Assume that* $1 < p < N, \mu + p < N, 1 < q < p^*$ *and let* $\Omega \subset \mathbb{R}^N$ *be an open Lipschitz domain such that one of the following holds:*

(i) Ω *is bounded and* $1 < q < p^*(\sigma, \mu)$.

(ii) *Ω has finite Lebergue measure, $0 \leq \sigma$ and $1 < q < p^*(\sigma, \mu)$.*

Consider the following problem

$$- div\Big(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{\mu}}\Big) = \lambda \frac{1}{|x|^{\sigma}}|u|^{q-2}u \ \text{ in } \ \Omega, \ u \in W^{1,p}_{\mu}(\Omega). \tag{4.20}$$

(1) If $q = p$, then (4.20) possesses a sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$.

(2) If $q > p$, then for any positive fixed λ, (4.20) *possess a sequence of solutions $\{v_n\}$ such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n \to \infty$ as $n \to \infty$.*

(3) If $q < p$, then for any positive fixed λ, (4.20) *possess a sequence of solutions $\{v_n\}$ such that $-\infty < c_1 \leq c_2 \leq \cdots \leq c_n \to 0$ as $n \to \infty$,*

where

$$c_n := \frac{1}{p}\int_{\Omega} \frac{|\nabla v_n|^p}{|x|^{\mu}}dx - \frac{1}{q}\lambda \int_{\Omega} \frac{|v_n|^q}{|x|^{\sigma}}dx.$$

Proof. By Theorem1.2, we only need to check that $V(x) = |x|^{-\sigma}$ satisfies the assumption (H). We first check the case (i): when Ω is bounded and $p < p^*(\sigma, \mu)$, we have $\sigma < \mu + p < N$, then we take $V(x) \equiv V_1(x) + V_2 = |x|^{-\sigma} + 0$ and choose $\eta = \sigma$ in the assumption (H), we see that $\frac{-p^*(\sigma,\mu)\sigma + q\sigma}{p^*(\sigma,\mu) - q} = -\sigma > -N$. Thus, $\big|V(x)\big|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}}|x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} \in L^1(\Omega)$. When Ω is bounded and $p^*(\sigma, \mu) \leq p$, we have that $\mu + p \leq \sigma$. By $q < p^*(\sigma, \mu)$, we have $\sigma < N - \frac{q}{p}(N - \mu - p) < N$. Hence, we take $V(x) \equiv V_1(x) + V_2(x) = |x|^{-\sigma} + 0$, recalling that $q < p^*(\sigma, \mu)$, we have

$$\frac{-p^*(\eta, \mu)\sigma + q\eta}{p^*(\eta, \mu) - q} > -N \Leftrightarrow \eta < N. \tag{4.21}$$

Then for any η satisfying $\frac{N\mu}{N-p} \leq \eta < \mu + p$, $\big|V(x)\big|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}}|x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} \in L^1(\Omega)$. The assumption (H) holds.

Next, we check for the case (ii): we prefer to introduce the characteristic function for any subset $A \subset \mathbb{R}^N$:

$$1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Case 1. If $p^*(\sigma, \mu) \leq p$, we take $V(x) \equiv V_1(x) + V_2(x)$ with

$$V_1(x) = |x|^{-\sigma}1_{\Omega \cap B_r(0)}(x), \quad V_2(x) = |x|^{-\sigma}1_{\Omega \cap B_r^c(0)}(x)$$

for some $r > 0$. Then similar to the arguments above, we have that

$$\big|V_1(x)\big|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}}|x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} \in L^1(\Omega),$$

since $q < \min\{p^*(\sigma,\mu), p^*\}$. Note that in this case, we have $\sigma \geq \mu + p$ and $p > q > 1$. It follows from $\sigma \geq 0$ that $-\sigma p + (\mu+p)q = -\sigma(p-q) - (\sigma - \mu - p)q \leq 0$. Hence

$$\lim_{R\to\infty} \int_{\{x\in\Omega:|x|>R\}} \big(V_2(x)\big)^{\frac{p}{p-q}} |x|^{\frac{(\mu+p)q}{p-q}} dx = 0 \tag{4.22}$$

as $R \to \infty$ due to the fact $L(\Omega) < \infty$. Hence, the assumption (H) is satisfied.

Case 2. If $p^*(\sigma,\mu) > p$, we have $0 \leq \sigma < \mu + p < N$ and $q < p^*(\sigma,\mu) \leq p^*(0,\mu)$. Now, take $V(x) \equiv V_1(x) + V_2(x)$ with

$$V_1(x) = |x|^{-\sigma} 1_{\Omega \cap B_r^c(0)}(x), \quad V_2(x) = |x|^{-\sigma} 1_{\Omega \cap B_r(0)}(x)$$

for some $r > 0$. For $V_1(x)$, take $\eta = \sigma$, we have

$$\big|V_1(x)\big|^{\frac{p^*(\eta,\mu)}{p^*(\eta,\mu)-q}} |x|^{\frac{q\eta}{p^*(\eta,\mu)-q}} = 1_{\Omega\cap B_r^c(0)}(x)|x|^{-\sigma} \in L^1(\Omega)$$

since $\sigma \geq 0$ and $L(\Omega) < \infty$. We also note that

(1) if $q < p$, V_2 satisfies (H_2) since $\sigma < \mu + p$.

(2) if $p \leq q < p^*(\sigma,\mu)$, we have $\bar{\sigma} := N - \frac{q}{p}(N - \mu - p) > \sigma$. Hence, V_2 satisfies (H_3).

Hence, the assumption (H) is also satisfied for the case of (ii). $\square$

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